AN ITERATIVE METHOD OF SOLVING ELASTO-PLASTIC PROBLEMS OF SOIL MECHANICS RELATING TO MORAINIC FOUNDATIONS

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An account is given of the theory of elasto-plastic problems in soil mechanics. Solution by the method of successive approximations combined with a network method is described.

Deformations in foundation soils are now found by applying the theory of linear deformations, which is valid only for a linear stress-strain relationship.

Experimental investigation of the deformation properties of various soils indicates that the dependence of strain on stress is generally nonlinear [1-3]. Elastic deformation is always accompanied by plastic deformation, which often exceeds the former by a factor of ten.

Accordingly, the deformations of foundation soils should, in general, be found by applying the elasto-plastic, not the linear theory.

I propose to use the theory of small elasto-plastic deformations to determine the deformations of morainic foundation soils due to external loading.

According to this theory, the method of solution of such problems reduces to the simultaneous examination of the statical and physical equations, the geometrical relationships, the expressions for stress and strain intensity, and the dependence of strain on stress. Thus, in solving problems in the theory of plasticity, at every point of the deformed body it is necessary to satisfy 18 equations, besides the boundary equations [4].

Il'yushin [5] has developed a special method, called the "method of elastic solutions," for solving problems involving the theory of small elasto-plastic deformations; it allows the plastic problem to be reduced to the successive solution of equations analogous to the Lamé equations in elasticity theory, with given boundary conditions. The solution of these equations, while fulfilling the boundary conditions, is very difficult, however. Therefore, in integrating the analogous equations in elasticity problems it is usual to employ the reverse method, assigning the displacements, as functions of the coordinates of the point, and on the basis of the boundary conditions determining the external forces acting at the surface of the body which the given displacements satisfy. The Saint-Venant method can also be used. In this case only part of the external forces and part of the displacements are assigned, and the remaining factors are found from the Lamé equations and the boundary conditions. Both these methods of solving elastic problems are inapplicable to the solution of problems of plasticity by the method of elastic solutions.

A simpler method of solving problems of plasticity is, I think, solution by the method of finite differences [6]. Essentially, this replaces the partial differential equations by partial difference equations, while the operator expressions linear with respect to derivatives correspond to expressions linear with respect to differences. As a result, the partial differential equations are replaced by a system of linear algebraic equations, in which the unknowns are values of the function at the nodes of the assumed type of network, the number of unknowns depending on the number of intermediate nodes of the network approximating the region being studied.

Equations of the Lamé type, which describe the elasto-plastic deformation of soils without taking volume forces into account, may be represented for points i, j, k (Fig. 1) in finite differences as follows:

$$\frac{k+1}{4}(v_{i+1,\,j-1,k}+v_{i-1,\,j+1,k}-v_{i-1,\,j-1,k}-v_{i+1,\,j+1,k}+$$
$$+w_{i+1,\,j,k-1}+w_{i-1,\,j,k+1}-w_{i-1,\,j,k-1}-w_{i+1,\,j,k+1})+$$
$$+(k+2)(u_{i+1,\,j,k}+u_{i-1,\,j,k}-2u_{i,j,k})+u_{i,j+1,k}+u_{i,j-1,k}+$$
$$+u_{i,j,k+1}+u_{i,j,k-1}-4u_{i,j,k}=\Delta h^2R_x;$$

$$\frac{k+1}{4}(u_{i+1,\,j-1,k}+u_{i-1,\,j+1,k}-u_{i-1,\,j-1,k}-u_{i+1,\,j+1,k}+$$
$$+w_{i,j+1,\,k-1}+w_{i,j-1,\,k+1}-w_{i,j-1,\,k-1}-w_{i,j+1,\,k+1})+$$

$$+ (k+2)(v_{i,j+1,k} + v_{i,j-1,k} - 2v_{i,j,k}) + v_{i+1,j,k} + v_{i-1,j,k} + $$
$$+ v_{i,j,k+1} + v_{i,j,k-1} - 4v_{i,j,k} = \Delta h^2 R_y;$$
$$\frac{k+1}{4}(u_{i+1,j,k-1} + u_{i-1,j,k+1} - u_{i-1,j,k-1} - u_{i+1,j,k+1} +$$
$$+ v_{i,j-1,k+1} + v_{i,j+1,k-1} - v_{i,j-1,k-1} - v_{i,j+1,k+1}) +$$
$$+ (k+2)(w_{i,j,k+1} + w_{i,j,k-1} - 2w_{i,j,k}) + w_{i+1,j,k} + w_{i-1,j,k} + \tag{1}$$
$$+ w_{i,j+1,k} + w_{i,j-1,k} - 4w_{i,j,k} = \Delta h^2 R_z,$$

where the following values are assumed for the first equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\Delta x^2}(u_{i+1,j,k} + u_{i-1,j,k} - 2u_{i,j,k});$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\Delta y^2}(u_{i,j+1,k} + u_{i,j-1,k} - 2u_{i,j,k});$$
$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\Delta z^2}(u_{i,j,k+1} + u_{i,j,k-1} - 2u_{i,j,k});$$
$$\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{4\Delta x \Delta y}(v_{i+1,j-1,k} + v_{i-1,j+1,k} - v_{i-1,j-1,k} - v_{i+1,j+1,k});$$
$$\frac{\partial^2 w}{\partial x \partial z} = \frac{1}{4\Delta x \Delta z}(w_{i+1,j,k-1} + w_{i-1,j,k+1} - w_{i-1,j,k-1} - w_{i+1,j,k+1});$$
$$R_x = \omega\left[\frac{1}{\Delta x^2}(u_{i+1,j,k} + u_{i-1,j,k} - 2u_{i,j,k}) + \right.$$
$$+ \frac{1}{\Delta y^2}(u_{i,j+1,k} + u_{i,j-1,k} - 2u_{i,j,k}) + \frac{1}{\Delta z^2}(u_{i,j,k+1} +$$
$$\left. + u_{i,j,k-1} - 2u_{i,j,k}) + \frac{1}{3}\left(\frac{\theta_{i+1,j,k} - \theta_{i,j,k}}{\Delta x}\right)\right] +$$
$$+ 2\left(\frac{\omega_{i+1,j,k} - \omega_{i,j,k}}{\Delta x}\right)\left[\left(\frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x}\right) - \frac{\theta_{i,j,k}}{3}\right] +$$
$$+ \left(\frac{\omega_{i,j+1,k} - \omega_{i,j,k}}{\Delta y}\right)\left[\left(\frac{u_{i,j+1,k} - u_{i,j,k}}{\Delta y}\right) + \right.$$
$$\left. + \left(\frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta x}\right)\right] + \left(\frac{\omega_{i,j,k+1} - \omega_{i,j,k}}{\Delta z}\right) \times$$
$$\times \left[\left(\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta z}\right) + \left(\frac{w_{i+1,j,k} - w_{i,j,k}}{\Delta x}\right)\right];$$
$$\Delta x = \Delta y = \Delta z = \Delta h;$$
$$\omega = 1 - \frac{\sigma_i}{3G\varepsilon_i}; \quad \theta = 3\varepsilon_m.$$

Values for the second and third equations of (1) may be obtained from the rule of circular permutation. At any point of the elasto-plastic region, in the solution by the method of elastic solutions the relation between stress component and strain components is expressed by

$$\sigma_x = \bar{\sigma}_x - 2G\omega(\varepsilon_x - \varepsilon_m); \qquad \tau_{xy} = \bar{\tau}_{xy} - G\omega\gamma_{xy};$$
$$\sigma_y = \bar{\sigma}_y - 2G\omega(\varepsilon_y - \varepsilon_m); \qquad \tau_{yz} = \bar{\tau}_{yz} - G\omega\gamma_{yz}; \tag{2}$$
$$\sigma_z = \bar{\sigma}_z - 2G\omega(\varepsilon_z - \varepsilon_m); \qquad \tau_{zx} = \bar{\tau}_{zx} - G\omega\gamma_{zx}.$$

Equations (2) are valid not only for internal points in the region examined, but also for edge points, the stress components at the edges necessarily conforming to the given boundary conditions. Equations (2) may be written in finite differences for points i, j, k (Fig. 2, a, b) as follows:

for a
$$\sigma_y^0 = \bar{\sigma}_y^0 - \frac{2G\omega}{3}\left(\frac{2v_{i,j+1,k} - 2v_{i,j,k}}{\Delta y} - \frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x} - \frac{w_{i,j,k+1} - w_{i,j,k}}{\Delta z}\right);$$

$$\tau_{xy}^0 = \bar{\tau}_{xy}^0 - G\omega\left(\frac{u_{i,j+1,k} - u_{i,j,k}}{\Delta y} + \frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta x}\right);$$

$$\tau_{yz}^0 = \bar{\tau}_{yz}^0 - G\omega\left(\frac{v_{i,j,k+1} - v_{i,j,k}}{\Delta z} + \frac{w_{i,j+1,k} - w_{i,j,k}}{\Delta y}\right);$$

for b
$$\sigma_x^0 = \bar{\sigma}_x^0 - \frac{2G\omega}{3}\left(\frac{2u_{i,j,k} - 2u_{i-1,j,k}}{\Delta x} - \frac{v_{i,j+1,k} - v_{i,j,k}}{\Delta y} - \frac{w_{i,j,k+1} - w_{i,j,k}}{\Delta z}\right); \tag{3}$$

$$\tau_{xz}^0 = \bar{\tau}_{xz}^0 - G\omega\left(\frac{w_{i,j,k} - w_{i-1,j,k}}{\Delta x} + \frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta z}\right).$$

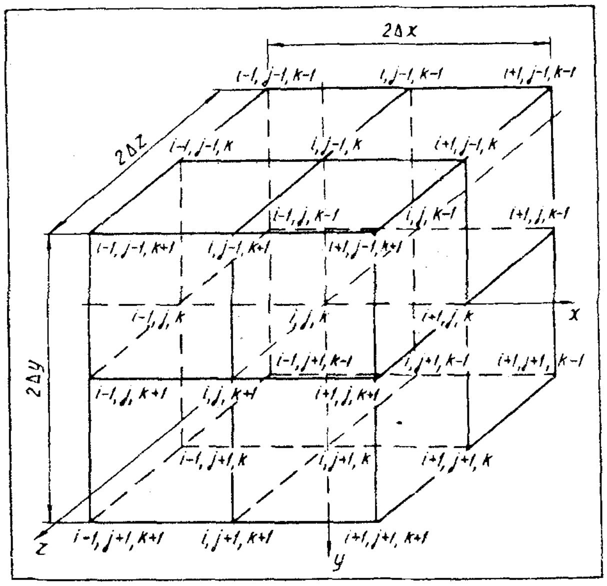

Fig. 1.

For points at the edge of the region examined, both equations (3) and the surface conditions must be satisfied:

$$P_{xv} = \bar{P}_{xv} + GR_{xv};$$
$$P_{yv} = \bar{P}_{yv} + GR_{yv}; \tag{4}$$
$$P_{zv} = \bar{P}_{zv} + GR_{zv},$$

where

$$R_{xv} = -\omega\left[\left(\frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x}\right)2l + \left(\frac{u_{i,j+1,k} - u_{i,j,k}}{\Delta y}\right)m + \left(\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta z}\right)n + \left(\frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta x}\right)m + \left(\frac{w_{i+1,j,k} - w_{i,j,k}}{\Delta x}\right)n\right].$$

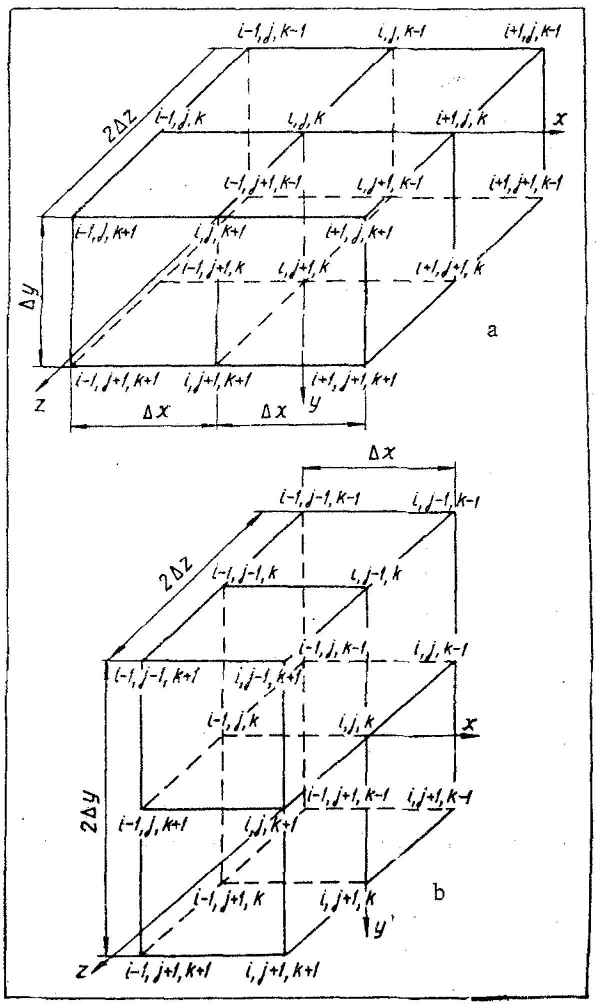

Fig. 2. Network for stress components at the edges: a – for points located on plane xoz, b – on plane yoz

The values of expressions R_{yv} and R_{zv} may be determined from the rule of circular permutation. Therefore, equating (3) to surface conditions (4), we obtain a new system of equations, relating the internal stresses in the region examined with the edge conditions.

In solving elasto-plastic problems by the method of elastic solutions, we assume $\omega = 0$ in the first approximation, i.e., we have the ordinary problem of the theory of elasticity, the solution of which, for given boundary conditions, may be obtained in closed form.

To solve elasto-plastic problems in the second approximation, it is first necessary to express the displacements of edge points of the network region in terms of displacements of the internal points adjacent to the edges and the boundary conditions, i.e., to include the unknown edge displacements in the overall iteration procedure.

We shall assume that at the surface of an elasto-plastic half-space a local load acts; this load can be resolved into a normal load σ_y^0 and tangential components τ_{xy}^0, τ_{yz}^0. Bounding the half-space by planes, and applying along these planes normal and tangential loads, corresponding to the internal stresses at the same points of the half-space, calculated from (4), we obtain the basic calculation scheme (Fig. 3). Fig. 3 shows the upper boundary points of the basic calculation scheme, and the internal network nodes adjacent to them. Loads acting along the edges of the network region are reduced to uniformly distributed loads relative to the edge node points of the network.

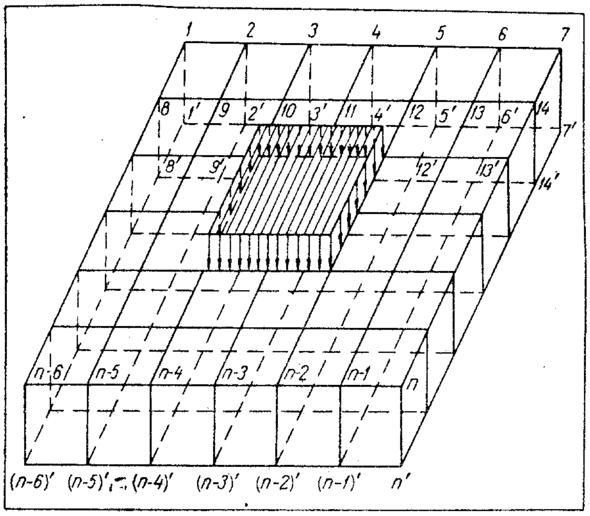

Fig. 3. Upper part of basic calculation scheme.

To construct the equations relating edge stresses to displacements of the edge points and displacements of the adjacent internal points (equations (3)), it is necessary to work around the network region systematically. The displacements of the corner points of the network region are expressed by the displacements of the adjoining edge points of the network.

To determine the displacements of the internal points of the assumed type of network, it is necessary, for each internal node point, to write down expressions in accordance with (1) for the components of the displacements as functions of the displacements of the adjacent points. The displacements of the edge points entering into these expressions must be represented as functions of the displacements of the adjacent internal points of the network and of the edge stresses, the edge displacements being included in the general iteration process.

Therefore, the problem of determining the displacements of the nodes of the network region reduces to the solution of a system of equations which can be represented in the canonical form:

$$\begin{aligned} &X_1a_{11} + X_2a_{12} + \ldots + X_na_{1n} + a_{1P} = 0; \\ &X_1a_{21} + X_2a_{22} + \ldots + X_na_{2n} + a_{2P} = 0; \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &X_1a_{n1} + X_2a_{n2} + \ldots + X_na_{nn} + a_{nP} = 0. \end{aligned} \tag{5}$$

In view of the considerable number of unknowns in (5), it is recommended that an electronic computer be used in the solution.

In solving problems in soil mechanics by the method of elastic solutions, sufficient accuracy can be obtained with the second approximation. If need be, however, this method may also be used, in conjunction with the network method, for solving elasto-plastic problems in the third and subsequent approximations. The method of solution of such problems is analogous to the method of solving elasto-plastic problems in the second approximation.

NOTATION

u, v, ω – displacement components; k – quotient of Lamé's constants λ and G; Δx, Δy, Δz – distances between nodes of network; σ_i – stress intensity; ε_i – strain intensity; ε_m – mean relative elongation; ε_x, ε_y, ε_z – relative elongations; γ_{xy}, γ_{xz}, γ_{zy} – relative shears; σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz} – stress components; $\overline{\sigma}_x$, $\overline{\sigma}_y$, $\overline{\sigma}_z$, $\overline{\tau}_{xy}$, $\overline{\tau}_{xz}$, $\overline{\tau}_{yz}$ – components of elastic stress; $\overline{P}_{x\nu}$, $\overline{P}_{y\nu}$, $\overline{P}_{z\nu}$ – surface loads for a perfectly elastic body; l, m, n – cosines of angles between corresponding axes and normal to area; a_{ik} – coefficients of equations; X_i – unknown quantities.

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